# The Cesàro Method for Spherical Trigonometry

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**Abstract**

In this essay, we give a brief introduction to spherical geometry and stereographic projection to derive spherical trigonometry formulas in a non-traditional way. In particular, we derive the spherical law of cosines and several of Delambre's Analogies and Napier's Analogies for spherical geometry (which in turn can be used to derive the hyperbolic counterparts, as explained in Appendix $A$).

## 1 Spherical Geometry

Spherical geometry is an area which is slowly disappearing from mainstream mathematics. In the early twentieth century, many high-school textbooks included spherical geometry, but today, even professional mathematicians are exposed to little spherical geometry [1]. Historically, spherical geometry was mainly studied for practical purposes in navigation and astronomy. Hence, in many books on spherical geometry, the formulas are adapted for logarithmic computations and many numerical calculation exercises are provided. We will consider spherical geometry from a purely mathematical perspective. All figures were produced using GeoGebra Classic 6.

**Definition 1.1.** The unit 2-sphere is defined as

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

We will refer to the unit 2-sphere as *the sphere* for brevity.

**Definition 1.2.** A spherical circle is the resulting circle along the intersection of a sphere and an arbitrary plane.

Certain spherical circles are called *great circles*, which we will frequently refer to as spherical lines.

**Definition 1.3.** A great circle (spherical line) is the resulting circle along the intersection of a sphere and a plane passing through the center of the sphere.

Since three points define a unique plane, two points on the sphere define a unique great circle. Great circles allow us to define a distance on the sphere.

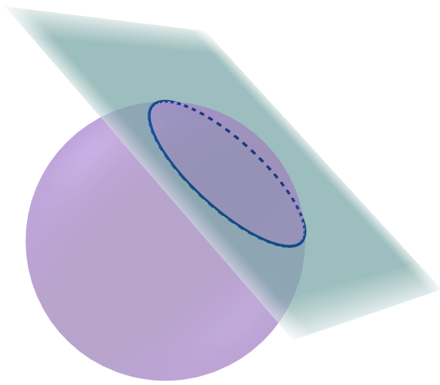


Figure 1: A Spherical Circle

**Definition 1.4.** The spherical distance between points $P, Q \in S^2$ is defined as the shortest distance along the great circle joining them.

It is not hard to show that this distance is indeed a metric. We denote the shortest great circle arc joining $A$ and $B$ by $\widehat{AB}$.

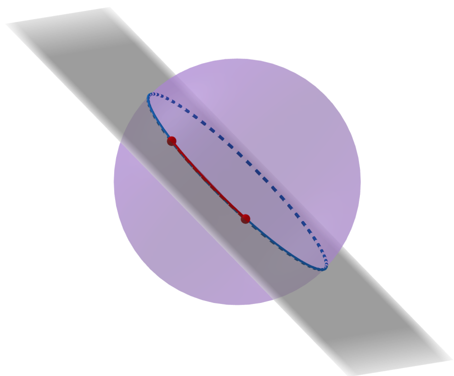


Figure 2: Distance Between Points On a Sphere

**Definition 1.5.** The spherical angle between two spherical lines is the acute angle between the planes that contain these arcs.

**Definition 1.6.** A spherical triangle $\triangle ABC$ is formed by three distinct points $A, B, C \in S^2$ and the three spherical lines joining them, which only intersect at the vertices. In the definition, we also specify the area enclosed by the spherical lines to avoid ambiguity.

*Remark.* The definition of a spherical quadrilateral -or a spherical polygon in general- is almost analogous to the definition above.

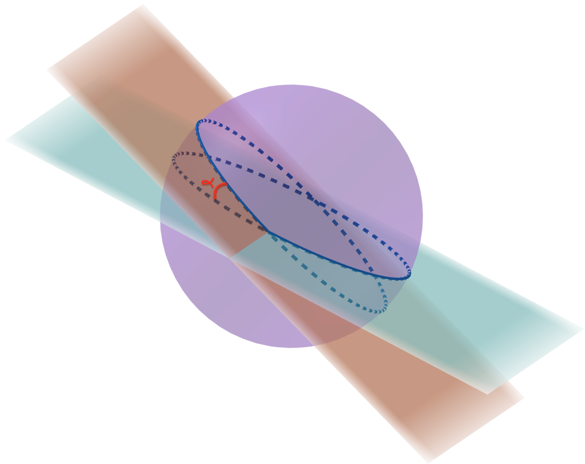


Figure 3: Angle Between Spherical Lines

A well-known theorem (which we state without proof) due to Albert Girard relates the radius of a sphere, the sum of the interior angles of a spherical triangle and the area of this spherical triangle.

**Theorem 1.1** (Girard's Theorem). *Let $\triangle ABC$ be a spherical triangle with side lengths $a, b, c$ and angle measures $\alpha, \beta, \gamma$. The surface area of $\triangle ABC$ equals,*

$$R^2(\alpha + \beta + \gamma - \pi)$$

*where $R$ denotes the radius of the sphere.*

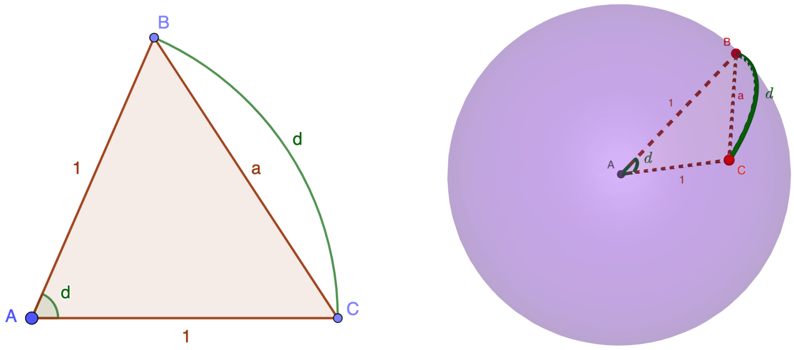
The expression $\alpha + \beta + \gamma - \pi$ is often referred to as the *spherical excess*, which we denote by $2E$ to prepare the notation for Section 3. We will only consider the unit sphere, in which case, the area of a spherical triangle equals its spherical excess.

*Remark.* We will often denote the angles and their measures using the same notation. Similarly, we will often confuse the names of line segments with their lengths. For example, we will write: A triangle with sides $a, b, c$ instead of a triangle with side lengths $a, b, c$.

For the next lemma, we will need the half-angle identities:

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos(\alpha)}{2} \quad \text{and} \quad \cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos(\alpha)}{2} \tag{1}$$

**Lemma 1.2** (Arc Length Lemma). *Consider a Euclidean triangle $\triangle ABC$ with $AB = AC = 1$ and $BC = a$, as given in Figure 4 (a). Let the length of the circular arc centered at $A$ joining $B$ and $C$ be $d$. Then, $a = 2\sin\left(\frac{d}{2}\right)$.*




(a) (b)

Figure 4: Arc Length Lemma

*Proof.* We note that $\measuredangle BAC = d$. Appyling the law of cosines to $\triangle ABC$ we get $a = \sqrt{2 - 2\cos d} = \sqrt{4\sin^2\left(\frac{d}{2}\right)} = 2\sin\left(\frac{d}{2}\right)$, where we use the half-angle identity for sine given in (1). $\square$

The Arc Length Lemma will be referred to frequently in section 3. As seen in Figure 4 $(b)$, we will use this lemma to convert the notion of length in Euclidean geometry to spherical geometry.

## 2 Stereographic Projection

There are several ways to project a sphere onto the plane. The advantages and disadvantages of each projection type is important in cartography (map-making). For example, the well-known Mercator projection distorts size and distance at high latitudes despite its many favourable properties. In this essay, we will use stereographic projection to continue our investigation of spherical geometry. One of the earliest uses of stereographic projection was seen in Hipparchus' ($\sim$ 125 BC) work on the astrolabe, an analog device which was used for astronomical calculations [6]. We first describe stereographic projection geometrically as seen in Figure 5.

**Definition 2.1.** A stereographic projection is a perspective projection from a point on the sphere onto a plane. Let the perspectivity point be the north pole, denoted by $P := (0, 0, 1)$. Take any point $A$ on $S^2 \backslash \{P\}$. The unique line passing through $P$ and $A$ intersects the plane $z = 0$ at exactly one point. This point is the stereographic projection of $A$.

We now describe stereographic projection algebraically.

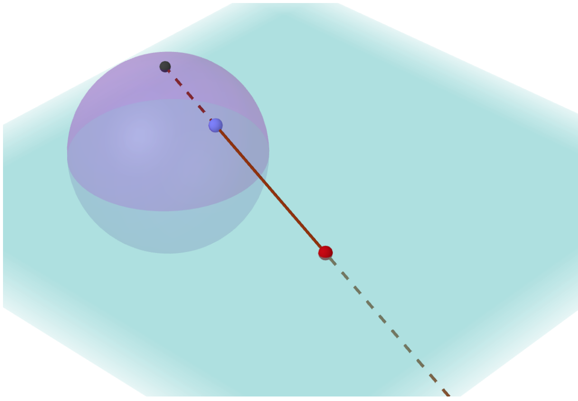


Figure 5: Stereographic Projection

**Theorem 2.1.** *The stereographic projection perspective from* $P$ *is a function* $f : S^2 \backslash \{P\} \to \mathbb{R}^2$ *given by,*

$$f(x,y,z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right) = (X,Y)$$

*and* $f$ *has an inverse function* $g : \mathbb{R}^2 \to S^2 \backslash \{P\}$ *given by*

$$g(X,Y) = \left( \frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2} \right) = (x,y,z).$$

*Proof.* Let $A = (x,y,z)$ with $x^2+y^2+z^2 = 1$. We can parametrise the line through $P$ and $A$ as $P + t(A-P) = (xt, yt, t(z-1)+1)$. The intersection of this line and $z = 0$ occurs when $t = \frac{1}{1-z}$. Substituting this value of $t$ in the equation of the line gives

$$(X,Y) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right).$$

Similarly, we can find the inverse of this map. Let $A = (X,Y,0)$. We can parametrise the line through $P$ and $A$ as $P + s(A-P) = (sX, sY, 1-s)$. The intersection of this line and $x^2+y^2+z^2 = 1$ occurs when $s = \frac{2}{1+X^2+Y^2}$. Substituting this value of $s$ in the equation of the line gives

$$(x,y,z) = \left( \frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2} \right).$$

□

*Remark.* This is a smooth and bijective function from $S^2 \backslash \{P\}$ onto the plane. Further, we note that this gives us a rational parametrisation of the sphere (except a point) compared to more common parametrisations of a sphere using spherical coordinates.

It is easy to see that stereographic projection does not preserve distances or areas. However, Theorems 2.2 and 2.3 will illustrate the two preservation properties of stereographic projection which we will use in the next section.

**Theorem 2.2.** *Stereographic projection maps spherical circles that don't pass through the projection point to circles on the plane. A spherical circle which passes through the projection point gets mapped to a line on the plane.*

*Proof.* By definition, a spherical circle is the intersection of a sphere and a plane (barring degenaricies). Hence it is the solution to the following system of equations:

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ ax + by + cz + d = 0 \end{cases}$$

Substituting $(x, y, z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2}\right)$ from Theorem 2.1 in the equation of the plane, we get

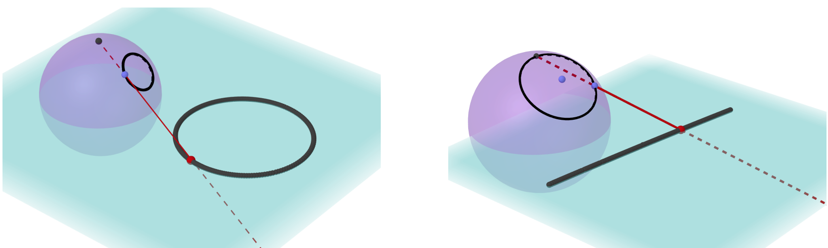
$$a\frac{2X}{X^2 + Y^2 + 1} + b\frac{2Y}{X^2 + Y^2 + 1} + c\frac{X^2 + Y^2 - 1}{X^2 + Y^2 + 1} + d = 0,$$

which is the same as

$$(c + d)X^2 + 2aX + (c + d)Y^2 + 2bY - c + d = 0.$$

We note that this equation represents a circle if $c + d \neq 0$, and a straight line otherwise.

□



(a) Circles to Circles

(b) Circles to Lines

Figure 6: Circle Mapping Property

The second important property of stereographic projection is the angle-preservation property. One of the earliest proofs of this property was discovered by Thomas Harriot in the late 16th century, although his proof was not published and was found later [7]. To understand this property, we will first need to define the Jacobian matrix.

**Definition 2.2.** Let $U \subset \mathbb{R}^n$ and let $f = (f_1, ..., f_n) : U \to \mathbb{R}^k$ be a continuously differentiable function. The Jacobian matrix of $f$ at $x$, denoted $Jf(x)$, is defined as

$$Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix},$$

where $(Jf)_{ij} = \frac{\partial f_i}{\partial x_j}$.

Computing the Jacobian for stereographic projection at $Q = (x, y, z) \in S^2$ we get

$$Jf(Q) = \frac{1}{1-z} \begin{pmatrix} 1 & 0 & \frac{x}{(1-z)} \\ 0 & 1 & \frac{y}{(1-z)} \end{pmatrix}.$$

**Theorem 2.3.** *Stereographic projection is angle-preserving (conformal).*

To be more precise, consider the tangent vectors $u, v$ to the sphere at a point $Q = (x, y, z) \in S^2$. Then the angle $\theta$ between $u$ and $v$ satisfies

$$\cos\theta = \frac{u \cdot v}{|u| \cdot |v|},$$

where $|\cdot|$ denotes the Euclidean norm. Our aim is to show that the action of $Jf$ on $u, v$ does not affect $\theta$. We will write $Jf(u)$ for $Jf(Q)(u)$ to relax the notation. We want to show,

$$\frac{u \cdot v}{|u| \cdot |v|} = \frac{Jf(u) \cdot Jf(v)}{|Jf(u)| \cdot |Jf(v)|} \tag{2}$$

holds. We now prove Theorem 2.3.

*Proof.* Let $Q = (x, y, z) \in S^2$. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be the tangent vectors to the sphere at $Q$. Then,

$$Jf(u) = \frac{1}{1-z} \begin{pmatrix} 1 & 0 & \frac{x}{(1-z)} \\ 0 & 1 & \frac{y}{(1-z)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{1-z} \begin{pmatrix} u_1 + \frac{x}{1-z} u_3 \\ u_2 + \frac{y}{1-z} u_3 \end{pmatrix}.$$

Let $Q$ be the position vector of point $Q$. Since $u$ is the tangent vector at $Q$ we have $Q \cdot u = 0$, which gives:

$$xu_1 + yu_2 + zu_3 = 0 \tag{3}$$

Using (3) and $x^2+y^2+z^2=1$, we observe:

$$(1-z)^2 Jf(u)\cdot Jf(v) = \begin{pmatrix} u_1 + \frac{x}{1-z}u_3 \\ u_2 + \frac{y}{1-z}u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 + \frac{x}{1-z}v_3 \\ v_2 + \frac{y}{1-z}v_3 \end{pmatrix}$$

$$= u_1v_1 + u_2v_2 + \frac{x^2+y^2}{(1-z)^2}u_3v_3 + \frac{1}{1-z}\left(u_3\left(xv_1+yv_2\right) + v_3\left(xu_1+yu_2\right)\right)$$

$$= u_1v_1 + u_2v_2 + \frac{1-z^2}{(1-z)^2}u_3v_3 - \frac{1}{1-z}\left(u_3\left(zv_3\right) + v_3\left(zu_3\right)\right)$$

$$= u_1v_1 + u_2v_2 + \frac{1+z}{1-z}u_3v_3 - \frac{2z}{1-z}u_3v_3$$

$$= u_1v_1 + u_2v_2 + u_3v_3$$

$$= u\cdot v$$

Once again, using (3) and $x^2+y^2+z^2=1$, we observe

$$\begin{aligned}(1-z)\left|Jf(u)\right| &= \left|\begin{pmatrix} u_1 + \frac{x}{1-z}u_3 \\ u_2 + \frac{y}{1-z}u_3 \end{pmatrix}\right| \\ &= \sqrt{u_1^2+u_2^2+\frac{x^2+y^2}{(1-z)^2}u_3^2 + \frac{2u_3}{1-z}\left(xu_1+yu_2\right)} \\ &= \sqrt{u_1^2+u_2^2+\frac{1-z^2}{(1-z)^2}u_3^2 - \frac{2u_3}{1-z}zu_3} \\ &= \sqrt{u_1^2+u_2^2+\frac{1+z}{1-z}u_3^2 - \frac{2z}{1-z}u_3^2} \\ &= \sqrt{u_1^2+u_2^2+u_3^2} \\ &= |u|.\end{aligned}$$

Repeating the same calculations for $v$ we also obtain $(1-z)\left|Jf(v)\right| = |v|$. The relations we obtained confirm (2), which concludes the proof.

□

*Remark.* Geometric proofs of Theorems 2.2 and 2.3 are given in [5].

## 3 The Cesàro Method

Giuseppe Cesàro (1849-1939) was an Italian professor of crystallography and mineralogy at the University of Liège. He was the brother of the mathematician Ernesto Cesàro. It is believed that Giuseppe Cesàro discovered "the Cesàro method" by applying stereographic projection on his work on crystallography [9]. The main reference for the Cesàro method is [3], which was written after the passing away of Cesàro by J. D. H. Donnay, a friend of Cesàro. Cesàro used the circle-preservation and conformality properties of stereographic projection to form a "bridge" between Euclidean geometry and spherical geometry.

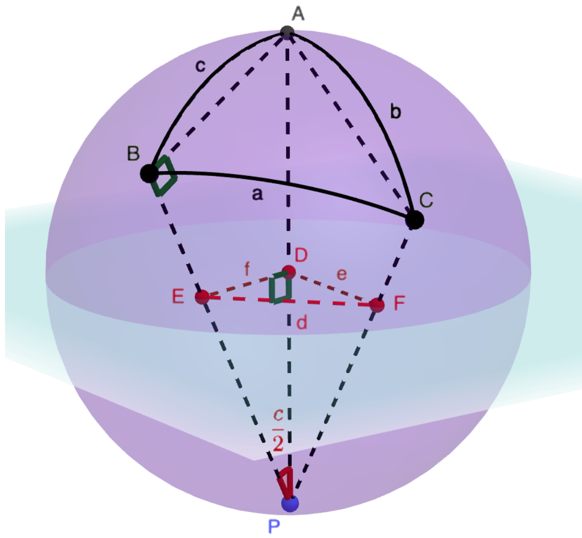



Figure 7: Sides of The Key Triangle

Given a spherical triangle $\triangle ABC$ with sides $a, b, c$ and angles $\alpha, \beta, \gamma$, place the vertex $A$ such that such that it coincides with the north pole, that is the point $(0, 0, 1)$. Consider a stereographic projection centered at the point $P := (0, 0, -1)$. Let $D, E, F$ be the stereographic projections of points $A, B, C$ respectively. Since the great circle through $A, B$ and the great circle through $C, A$ both pass through the perspectivity point, by Theorem 2.2, arcs $\widehat{AB}$ and $\widehat{CA}$ will be projected to straight line segments $DE$ and $FD$. However, the great circle through $B, C$ does not pass through the perspectivity point, so $\widehat{BC}$ will be projected to the circular arc connecting $E$ and $F$. Connecting $E$ and $F$ with a straight line segment, we obtain the triangle $\triangle DEF$. This triangle is referred to as "Cesàro's triangle of elements" in [3], but we will refer to it as *the key triangle* for short. Before we prove the properties of the key triangle, we will state, without proof, a theorem (often taught in high-schools) relating central angles and inscribed angles.

**Theorem 3.1** (The Inscribed Angle Theorem)**.** *The measure of an inscribed angle on a circle is half the measure of the central angle which subtends the same arc.*

This theorem allows us to conclude $\measuredangle APB = \frac{c}{2}$ in Figure 7 since $\measuredangle ADB = c$. Let $d, e, f$ be the sides of $\triangle DEF$.

**Theorem 3.2.** *The sides of the key triangle are given by* $e = \tan\left(\frac{b}{2}\right)$, $f = \tan\left(\frac{c}{2}\right)$, *and* $d = \sin\left(\frac{a}{2}\right) \sec\left(\frac{b}{2}\right) \sec\left(\frac{c}{2}\right)$.

*Proof.* Throughout the proof, we use the fact that $|DP| = 1$ and $|AP| = 2$ since we are working over the unit sphere. By the Inscribed Angle Theorem, $\measuredangle APB = \frac{c}{2}$. Noting $\triangle DEP$ is a right-angled triangle we get $f = \tan\left(\frac{c}{2}\right)$. Similarly, we get $e = \tan\left(\frac{b}{2}\right)$. For the remaining side $d$, we note that $\triangle BAP$ is also a right-angled triangle (by the inscribed angle theorem), which gives $\triangle DEP \sim \triangle BAP$. By symmetry, it follows that $\triangle DFP \sim \triangle CAP$. Combining the ratio proportionalities we obtained from these pair of similar triangles yields $\triangle PBC \sim \triangle PFE$. Thus,

$$\frac{|FE|}{|BC|} = \frac{|PF|}{|PB|} \iff \frac{d}{2\sin\left(\frac{a}{2}\right)} = \frac{\sec\left(\frac{b}{2}\right)}{2\cos\left(\frac{c}{2}\right)}$$

where $|BC| = 2\sin\left(\frac{a}{2}\right)$ follows from the Arc Length Lemma and the expressions for $|PF|, |PB|$ follow from the fact that $\triangle DFP$ and $\triangle BAP$ are right-angled triangles. Rearranging gives the result. □

Let $\delta, \epsilon, \phi$ be the angles of $\triangle DEF$ and let $2E$ denote the area of the spherical triangle $\triangle ABC$. From Girard's Theorem, we recall that $2E = \alpha + \beta + \gamma - \pi$.

**Theorem 3.3.** *The angles of the key triangle are given by* $\delta = \alpha, \epsilon = \beta - E$, *and* $\phi = \gamma - E$.

*Proof.* Consider the tangent lines to the arc connecting $E$ and $F$ at points $E$ and $F$, as given in Figure 8. Let $T$ be the intersection of these tangent lines. Since stereographic projection is conformal we have $\delta = \alpha$, $\measuredangle DET = \beta$ and $\measuredangle DFT = \gamma$. Considering the quadrilateral $DETF$ we note $\measuredangle ETF = 2\pi - (\alpha + \beta + \gamma) = 2\pi - (\pi + 2E) = \pi - 2E$. Since $TE$ and $TF$ are both tangents we have $TE = TF$, which implies $\measuredangle FET = \measuredangle TFE = E$, and it follows that $\epsilon = \beta - E$ and $\phi = \gamma - E$. □

Equipped with the properties of the key triangle, we are now ready to prove the spherical law of cosines. We make note of the following trigonometric identities, which we will use frequently.

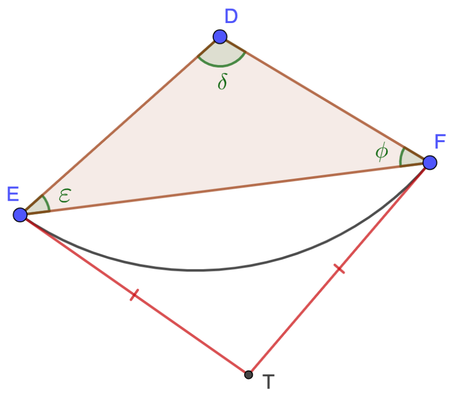



Figure 8: Angles of The Key Triangle

The double angle identities:

$$\sin(\alpha) = 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) \quad \text{and} \quad \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{4}$$

The phase shift identities:

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha) \quad \text{and} \quad \tan\left(\frac{\pi}{2} - \alpha\right) = \cot(\alpha) \tag{5}$$

Angle sum/difference identities:

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\ \cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \end{aligned} \tag{6}$$

Angle sum/difference to product identities for sine:

$$\sin(\alpha) \pm \sin(\beta) = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{7}$$

**Theorem 3.4** (The Spherical Law of Cosines)**.** *Let $\triangle ABC$ be a spherical triangle with sides $a, b, c$ and angles $\alpha, \beta, \gamma$. Then,*

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

*Proof.* Applying the planar law of cosines to the key triangle gives

$$d^2 = e^2 + f^2 - 2ef\cos\delta.$$

Substituting from Theorems 3.2 and 3.3 we get

$$\sin^2\left(\frac{a}{2}\right)\sec^2\left(\frac{b}{2}\right)\sec^2\left(\frac{c}{2}\right) = \tan^2\left(\frac{b}{2}\right) + \tan^2\left(\frac{c}{2}\right) - 2\tan\left(\frac{b}{2}\right)\tan\left(\frac{c}{2}\right)\cos\alpha.$$

Multiplying both sides by $\cos^2\left(\frac{b}{2}\right)\cos^2\left(\frac{c}{2}\right)$ we get that $\sin^2\left(\frac{a}{2}\right)$ equals

$$\sin^2\left(\frac{b}{2}\right)\cos^2\left(\frac{c}{2}\right)+\sin^2\left(\frac{c}{2}\right)\cos^2\left(\frac{b}{2}\right)-2\sin\left(\frac{b}{2}\right)\cos\left(\frac{b}{2}\right)\sin\left(\frac{c}{2}\right)\cos\left(\frac{c}{2}\right)\cos\alpha.$$

After applying the half-angle identities given in (1), expanding, and simplifying we arrive at the result,

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha. \qquad \square$$

Setting $\alpha = \frac{\pi}{2}$ yields the Pythagorean Theorem for spherical triangles.

**Corollary** (The Spherical Pythagorean Theorem)**.** *Let $\triangle ABC$ be a spherical triangle with sides $a, b, c$ and angles $\alpha, \beta, \gamma$, where $\alpha$ is a right angle. Then,*

$$\cos a = \cos b\cos c.$$

Similarly, we can apply the planar law of sines to the key triangle. However, it is not obvious how the resulting expression relates to the spherical law of sines, but it yields two of the four Delambre's Analogies [1] (also known as Gauss' Analogies). We note that there was some confusion over history on who discovered these analogies due to an incorrect reference given by Delambre in one of his works. It seems that Delambre, Mollweide, and Gauss discovered these analogies in similar time periods, with Delambre coming first [4]. For the proof of Delambre's Analogies we will need a simple lemma about proportions whose proof we omit since it follows from simple algebraic manipulations.

**Lemma 3.5.** *Let $a, b, c, d \in \mathbb{R}$ and $b, d, b \pm d \neq 0$. If $\frac{a}{b} = \frac{c}{d}$ it follows that:*

*i.* $\frac{a}{b} = \frac{a+c}{b+d}$

*ii.* $\frac{a}{b} = \frac{a-c}{b-d}$

**Theorem 3.6** (Delambre's Second and Fourth Analogies)**.** *Let $\triangle ABC$ be a spherical triangle with sides $a, b, c$ and angles $\alpha, \beta, \gamma$. Then,*

$$\frac{\cos\left(\frac{\beta-\gamma}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{b+c}{2}\right)}{\sin\left(\frac{a}{2}\right)} \quad \textit{and} \quad \frac{\sin\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{b-c}{2}\right)}{\sin\left(\frac{a}{2}\right)}.$$

*Proof.* Applying the law of sines to the key triangle gives

$$\frac{d}{\sin\delta} = \frac{e}{\sin\varepsilon} = \frac{f}{\sin\phi}.$$

[1] The word *analogies* is archaic for *proportions*.

Substituting from Theorems 3.2 and 3.3 we get

$$\frac{\sin\left(\frac{a}{2}\right)\sec\left(\frac{b}{2}\right)\sec\left(\frac{c}{2}\right)}{\sin\alpha} = \frac{\tan\left(\frac{b}{2}\right)}{\sin(\beta - E)} = \frac{\tan\left(\frac{c}{2}\right)}{\sin(\gamma - E)}.$$

Multiplying the whole equality by $\cos\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)$ yields

$$\frac{\sin\left(\frac{a}{2}\right)}{\sin\alpha} = \frac{\sin\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)}{\sin(\beta - E)} = \frac{\cos\left(\frac{b}{2}\right)\sin\left(\frac{c}{2}\right)}{\sin(\gamma - E)}. \tag{8}$$

Applying Lemma 3.5 $(i)$ to (8) gives

$$\frac{\sin\left(\frac{a}{2}\right)}{\sin\alpha} = \frac{\sin\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right) + \cos\left(\frac{b}{2}\right)\sin\left(\frac{c}{2}\right)}{\sin(\beta - E) + \sin(\gamma - E)}.$$

We use the double angle identity for sine given in (4) for the denominator of the left-hand side, and the angle sum identity for sine given in (6) for the numerator of right-hand side. Further, using the sum to product identity for sine given in (7) we can rewrite the denominator of the right-hand side, which yields

$$\frac{\sin\left(\frac{a}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{b+c}{2}\right)}{2\sin\left(\frac{\beta+\gamma-2E}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right)}.$$

Further, recalling $\alpha + \beta + \gamma - \pi = 2E$, we get $\sin\left(\frac{\beta+\gamma-2E}{2}\right) = \sin\left(\frac{\pi-\alpha}{2}\right) = \cos\left(\frac{\alpha}{2}\right)$, where we use the phase shift identity for sine given in (5). Thus, we obtain

$$\frac{\sin\left(\frac{a}{2}\right)}{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{b+c}{2}\right)}{2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right)}.$$

which upon simplifying and rearranging gives Delambre's Second Analogy. The fourth analogy follows similarly by applying Lemma 3.5 $(ii)$ to (8) and using the difference to product identity for sine given in (7).

$\square$

There are one more set of analogies, which are crucial for spherical trigonometry: Napier's Analogies, named after the Scottish mathematician John Napier (1550 – 1617), more famously known as the person who discovered the logarithm. We will derive the second of the four analogies Napier discovered.

**Theorem 3.7** (Napier's Second Analogy)**.** *Let $\triangle ABC$ be a spherical triangle with sides $a, b, c$ and angles $\alpha, \beta, \gamma$. Then,*

$$\frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\cot\left(\frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{b-c}{2}\right)}{\sin\left(\frac{b+c}{2}\right)}.$$

For the proof we will need a relatively unknown Euclidean geometry theorem.

**Lemma 3.8** (Law of Tangents)**.** *Let $\triangle ABC$ be a Euclidean triangle with sides $a, b, c$ and angles $\alpha, \beta, \gamma$. Then,*

$$\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} = \frac{a-b}{a+b}.$$

*Proof.* By the law of sines, we know that $\frac{a}{\sin\alpha} \overset{(\star)}{=} \frac{b}{\sin\beta}$. Applying Lemma 3.5 to $(\star)$ we obtain

$$\frac{a+b}{\sin\alpha+\sin\beta} = \frac{a-b}{\sin\alpha-\sin\beta}$$

and the result follows from the trigonometric identity

$$\tan\left(\frac{\alpha\pm\beta}{2}\right) = \frac{\sin\alpha\pm\sin\beta}{\cos\alpha+\cos\beta}. \qquad \square$$

We can now prove Napier's Second Analogy.

*Proof.* Applying the law of tangents to the sides $e$ and $f$ of the key triangle gives

$$\frac{\tan\left(\frac{\varepsilon-\phi}{2}\right)}{\tan\left(\frac{\varepsilon+\phi}{2}\right)} = \frac{e-f}{e+f}.$$

Substituting from Theorems 3.2 and 3.3 we get

$$\frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma-2E}{2}\right)} = \frac{\tan\left(\frac{b}{2}\right)-\tan\left(\frac{c}{2}\right)}{\tan\left(\frac{b}{2}\right)+\tan\left(\frac{c}{2}\right)}.$$

We can simplify the right hand side by multiplying the numerator and denominator by $\cos\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)$ and using the angle sum/difference identities for sine given in (6). Further, recalling $\alpha+\beta+\gamma-\pi = 2E$ and using the phase shift identity for tangent given in (5) we can also simplify the denominator of the left hand side. Thus,

$$\frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\cot\left(\frac{\alpha}{2}\right)} = \frac{\sin\left(\frac{b-c}{2}\right)}{\sin\left(\frac{b+c}{2}\right)}. \qquad \square$$

*Remark.* The remaining analogies of Delambre and Napier can be derived by an extension of the Cesàro method and are given in [3].

*Remark.* For all the spherical trionometry formulas we have derived so far, mnemonics are given in [1], which provides a way to remember all of them.

## 4 Concluding Remarks

From an educational perspective, the Cesàro method has received several different opinions. For example, one of the reviewers of *Spherical Trigonometry After the Cesàro Method* praises the book in [2]:

"Thus if the Cesàro route is selected instead of the usual procedure, the student will not only have a more interesting trip through the subject but he will gain more in mathematical maturity-and mathematical maturity makes up perhaps a major portion of the profit derived from studying mathematics."

On the other hand, this method might not be suitable for the student who is only interested in the practicality of spherical geometry. We have indeed spent significant effort to develop the main properties of stereographic projection and the properties of the key triangle, which might feel overwhelming.

While both views have valid points, we can conclude that this method contributes to the richness of the subject by providing us with a non-traditional method of gaining intuition on spherical geometry.

## References


[1] John Conway and Alex Ryba. Remembering spherical trigonometry. *The Mathematical Gazette*, 100(547):1–8, 2016. ISSN 00255572. URL `http://www.jstor.org/stable/44161700`.

[2] H. V. Craig. *The American Mathematical Monthly*, 53(1):32–33, 1946. ISSN 00029890, 19300972. URL `http://www.jstor.org/stable/2306085`.

[3] J. D. H. (Joseph Désiré Hubert) Donnay. *Spherical trigonometry after the Cesàro method [by] J. D. H. Donnay.* Interscience publishers, inc., New York, N. Y, 1945.

[4] I. Todhunter M.A. F.R.S. Xii. note on the history of certain formulæ in spherical trigonometry. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 45(298):98–100, 1873. doi: 10.1080/14786447308640820. URL `https://doi.org/10.1080/14786447308640820`.

[5] David Hilbert. *Geometry and the imagination.* AMS Chelsea Pub, Providence, R.I, 2nd edition, 1999. ISBN 9780821819982;0821819984;.

[6] Richard J. Howarth. History of the stereographic projection and its early use in geology. *Terra Nova*, 8(6):499–513, 1996. doi: https://doi.org/10.1111/j.1365-3121.1996.tb00779.x. URL `https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1365-3121.1996.tb00779.x`.

[7] J. A. Lohne. Essays on thomas harriot. *Archive for History of Exact Sciences*, 20(3):189–312, Sep 1979. ISSN 1432-0657. doi: 10.1007/BF00327737. URL `https://doi.org/10.1007/BF00327737`.

[8] John G. Ratcliffe. *Foundations of hyperbolic manifolds*. Springer, Cham, Switzerland, third edition, 2019. ISBN 9783030315979;3030315975;.

[9] Glen Van Brummelen. *Heavenly mathematics: the forgotten art of spherical trigonometry*. Princeton University Press, Princeton, 2013. ISBN 1400844800;9781400844807;.

# A Hyperbolic Trigonometry

Hyperbolic geometry is another non-Euclidean geometry in which there exists infinitely many parallels to a line passing through a point which does not lie on the line. Although hyperbolic geometry is not the subject of this essay, we note that the "duality" between spherical geometry and hyperbolic geometry can be used to gain new insights into hyperbolic geometry; more concretely, we can derive the hyperbolic trigonometry formulas from their spherical counterparts. For example, let us state the hyperbolic law of cosines to notice how similar it looks to the spherical law of cosines.

**Theorem A.1** (The Hyperbolic Law of Cosines). *Let $\triangle ABC$ be a hyperbolic triangle with sides $a, b, c$ and angles $\alpha, \beta, \gamma$. Then,*

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha.$$

The reason for this similarity is that, as stated in [8, p. 83], the identities $\sin(ia) = i \sinh a$ and $\cos(ia) = \cosh a$ allow us to deduce formulas for hyperbolic trigonometry from spherical trionometry. Replacing the sides $a, b, c$ by $ia, ib, ic$ in spherical trigonometry formulas yields the hyperbolic analogues of these formulas. Likewise, the reader can obtain the hyperbolic analogues of the remaining spherical trigonometry theorems proven in this essay.