Complex Bashing

Every year, over 100 countries participate in the International Mathematical Olympiad (IMO), and classical Euclidean geometry is an important area included in the IMO. The technique we will describe in this article is often informally called "complex bashing" in Olympiad terms, which stands for bashing geometry with complex numbers. It is an alternative method to solve geometry problems without using elementary geometry. Complex numbers have two key properties that make them useful in geometry: first, complex numbers inherit certain properties of vectors, and second, the multiplication of complex numbers can be used to describe rotation in the plane. Our main objective in this article is to develop a number of propositions that are useful for solving geometry problems with complex numbers. We will then go over an example problem in detail. We first state a series of obvious facts in a proposition without proof. This proposition will lay the foundation for all of our work.

Proposition 1. *Let z and w be complex numbers. Then, the following are true:*

(i) $z = \overline{z}$ *if and only if z is a real number.*

(ii) $z = -\overline{z}$ *if and only if z is purely imaginary.*

(iii) $|z|^2 = z\overline{z}$

(iv) *For any two complex numbers, conjugation is distributive over addition, subtraction, multiplication and division.*

(v) $\arg(zw) = \arg(z) + \arg(w)$

We note that for $|z| = 1$, **(iii)** becomes $\overline{z} = \frac{1}{z}$. For our purposes, this is a crucial equality since most proofs using complex numbers involve the unit circle in the complex plane. As an example for **(iv)**, the following equality holds for the complex numbers a,b,c,d:

$$\overline{\left(\frac{a+b}{c+d}\right)} = \frac{\overline{a}+\overline{b}}{\overline{c}+\overline{d}}$$

We will follow the convention of using lower case letters to denote the affixes of complex numbers. For example, O denotes the origin of the complex plane which corresponds to the complex number o. We will write XY to denote the line passing through points X and Y unless otherwise stated.

Proposition 2. *Let A,B,C,D be distinct points. Then, $AB \perp CD$ if and only if the following holds:*

$$\frac{d-c}{b-a} = -\overline{\left(\frac{d-c}{b-a}\right)}$$

Sometimes it is more convenient to write this as:

$$\frac{d-c}{b-a} + \overline{\left(\frac{d-c}{b-a}\right)} = 0$$

Proof. We want to show that $AB \perp CD$ if and only if $\frac{d-c}{b-a}$ is purely imaginary. We can think of $(d-c)$ and $(b-a)$ as vectors. From Proposition 1(v), we deduce that $(d-c)$ and $(b-a)$ are perpendicular when either

$$\left(\frac{d-c}{b-a}\right) \equiv \frac{\pi}{2} \pmod{2\pi}$$

or

$$\left(\frac{d-c}{b-a}\right) \equiv \frac{3\pi}{2} \pmod{2\pi}$$

which is equivalent to saying that $\frac{d-c}{b-a}$ is purely imaginary. ∎

Proposition 3. *The points A,B,C are collinear if and only if the following holds:*

$$\frac{c-a}{c-b} = \overline{\left(\frac{c-a}{c-b}\right)}$$

Proof. The proof is very similar to that of Proposition 2, so we omit it. ∎

Proposition 4. *Let A,B,C,D be distinct points that lie on the unit circle. Then, $ad + bc = 0$ if and only if $AD \perp BC$.*

Proof. Assume $AD \perp BC$. Since a,b,c,d lie on the unit circle, we have:

$$\bar{a} = \frac{1}{a},\ \bar{b} = \frac{1}{b},\ \bar{c} = \frac{1}{c},\ \bar{d} = \frac{1}{d}$$

Combining these with Proposition 2, we get:

$$\begin{aligned} 0 &= \frac{d-a}{b-c} + \overline{\left(\frac{d-a}{b-c}\right)} \\ &= \frac{d-a}{b-c} + \frac{\bar{d}-\bar{a}}{\bar{b}-\bar{c}} \\ &= \frac{d-a}{b-c} + \frac{\frac{1}{d}-\frac{1}{a}}{\frac{1}{b}-\frac{1}{c}} \\ &= -\frac{(a-d)(ad+bc)}{ad(b-c)} \end{aligned}$$

Since $a \neq d$, it follows that $ad+bc=0$, as desired. The other direction of the proof can be obtained by reversing this argument. ∎

Proposition 5. *Let A and B be distinct points on the unit circle. Let C be the foot of the perpendicular from an arbitrary point D to the line AB. Then:*

$$c = \frac{a+b+d-ab\bar{d}}{2}$$

Proof. Since a and b lie on the unit circle, we have:

$$\bar{a} = \frac{1}{a},\ \bar{b} = \frac{1}{b}$$

We know that A,B,C are collinear, so we can use Proposition 3:

$$\frac{c-a}{c-b} = \overline{\left(\frac{c-a}{c-b}\right)} = \frac{\bar{c}-\bar{a}}{\bar{c}-\bar{b}} = \frac{\bar{c}-\frac{1}{a}}{\bar{c}-\frac{1}{b}}$$

The other condition we have is $AB \perp CD$. Thus, by Proposition 2, we also get:

$$\frac{d-c}{b-a} = -\overline{\left(\frac{d-c}{b-a}\right)} = -\frac{\bar{d}-\bar{c}}{\frac{1}{b}-\frac{1}{a}}$$

Thus, we need to solve the following systems of equations:

$$\begin{cases} \dfrac{c-a}{c-b} = \dfrac{\bar{c}-\frac{1}{a}}{\bar{c}-\frac{1}{b}} \\ \dfrac{d-c}{b-a} = -\dfrac{\bar{d}-\bar{c}}{\frac{1}{b}-\frac{1}{a}} \end{cases}$$

Solving the system for c gives:

$$c = \frac{a+b+d-ab\bar{d}}{2}$$

∎

Remark. As seen in Figure 1, point C need not lie within the line segment AB.

We are now ready to attempt a problem.

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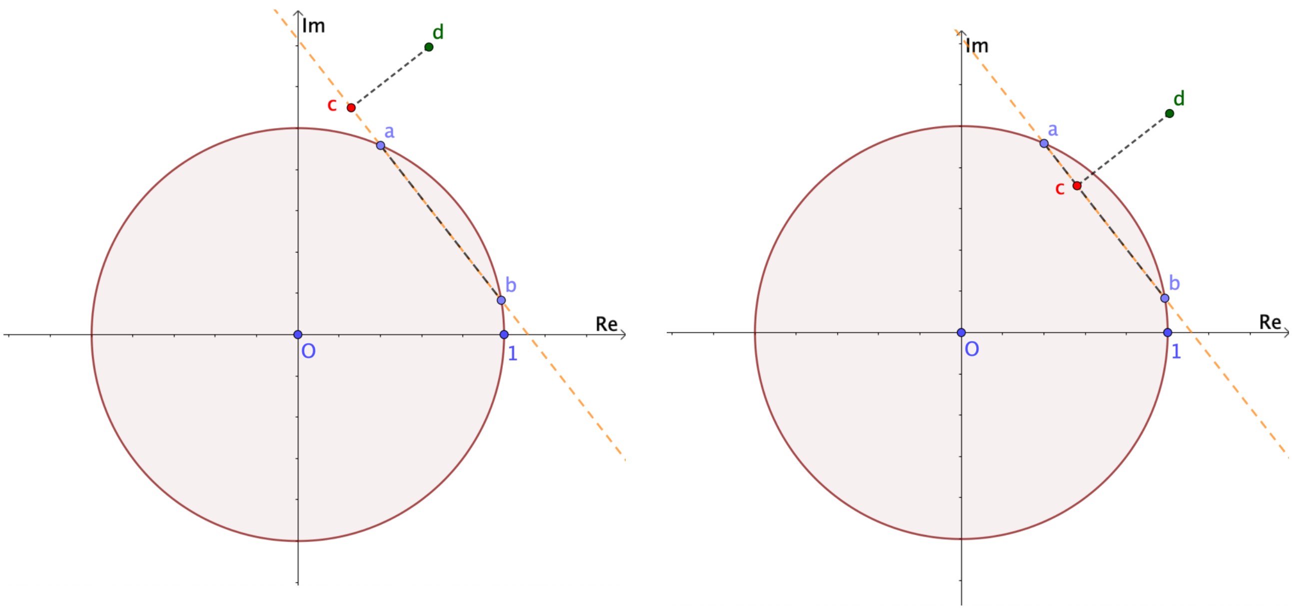

Figure 1. *Foot of the Perpendicular*

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Problem. *In ΔABC, the line passing through B is perpendicular to the line AC at E. This line meets the circumcircle of ΔABC at D. The foot of the perpendicular from D to the line BC is F. If O is the centre of the circumcircle of ΔABC, prove that OB is perpendicular to EF.*

(Turkey TST 1992 P2)

The problem we will solve is from the 1992 team selection test for the Turkish IMO team. This is the second problem (P2) of the paper and is a fairly simple P2 by today's standards. However, to an untrained eye, it can be difficult to solve this problem with elementary geometry. Before continuing with the proof, the reader is encouraged to try to produce a solution with elementary geometry to get a feel for the problem.

Before starting the proof, we give a sketch of the problem in Figure 2. We see that F may or may not lie inside the line segment BC depending on ΔABC.

Proof. Without loss of generality, assume that the circumcircle of ΔABC is the unit circle. This is an extremely common technique that is used while working with complex numbers. Then, a,b,c,d lie on the unit circle, so we have:

$$\overline{a} = \frac{1}{a},\ \overline{b} = \frac{1}{b},\ \overline{c} = \frac{1}{c},\ \overline{d} = \frac{1}{d}$$

We want to show $OB \perp EF$. By Proposition 2, this is equivalent to showing the following expression is equal to zero:

$$\frac{b-o}{f-e} + \overline{\left(\frac{b-o}{f-e}\right)} \tag{※}$$

Using Proposition 5, we can get expressions for e and f:

$$e = \frac{a+c+b-\frac{ac}{b}}{2}$$

$$f = \frac{b+c+d-\frac{bc}{d}}{2}$$

Let us first compute $f-e$ by substituting for e and f:

$$f-e = \frac{bd^2 - abd - b^2c + acd}{2bd}$$

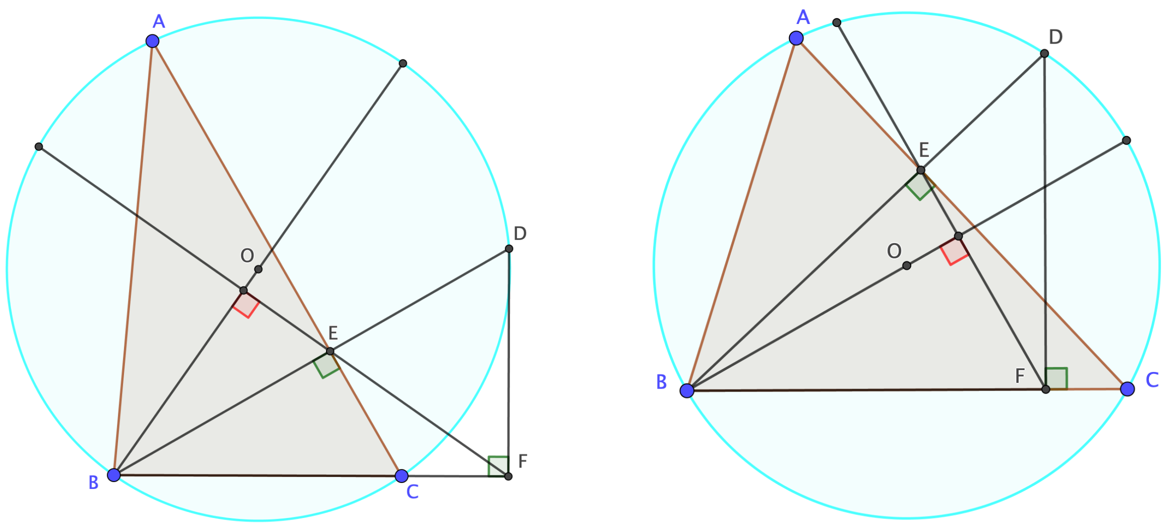

Figure 2. *Diagrams for the Problem*

Noting that a, b, c, d all lie on the unit circle, we can also compute $\overline{f-e}$:

$$\begin{aligned}
\overline{f-e} &= \frac{\overline{bd^2} - \overline{abd} - \overline{b^2c} + \overline{acd}}{\overline{2bd}} \\
&= \frac{\frac{1}{bd^2} - \frac{1}{abd} - \frac{1}{b^2c} + \frac{1}{acd}}{\frac{2}{bd}} \\
&= \frac{abc - ad^2 + b^2d - bcd}{2abcd}
\end{aligned}$$

Our aim was to show the expression in (※) is equal to 0. Substituting all we have obtained into this expression:

$$\begin{aligned}
\frac{b-o}{f-e} + \overline{\left(\frac{b-o}{f-e}\right)} &= \frac{b}{f-e} + \frac{\overline{b}}{\overline{f-e}} \\
&= \frac{b}{\frac{bd^2-abd-b^2c+acd}{2bd}} + \frac{\frac{1}{b}}{\frac{abc-ad^2+b^2d-bcd}{2abcd}} \\
&= -\frac{2d(b-c)\left(b^2-ad\right)(ac+bd)}{\left(abc-ad^2+b^2d-bcd\right)\left(abd-acd+b^2c-bd^2\right)}
\end{aligned}$$

Since a, b, c, d all lie on the unit circle, we know $ac + bd = 0$ by Proposition 4. Therefore, the fraction vanishes, as desired.

Remark. We didn't have to make any geometric observations to solve this problem. We merely applied the propositions we derived, and after some algebra, we arrived at the result. ■

We conclude with a warning: complex numbers should not be the first tool in a geometer's toolbox.

As we saw in the problem above, complex numbers lead to tedious calculations. Hence, it is very important to understand that a clever geometric observation can drastically simplify calculations, and this observation could be enough to solve the problem without using complex numbers at all!

For the interested reader who would like to learn more about complex bashing in geometry, I recommend chapters 3 and 4 of the book *Complex Numbers from A to ... Z* by Titu Andreescu and Dorin Andrica.

– Doğukan E. Türköz

All diagrams used in this article produced using GeoGebra