On The Properties of The Commuting Conjugacy Class Graph of The Symmetric Group

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Abstract

For a group G, the commuting conjugacy class graph of G is the graph $\Gamma(G)$ with vertices being the non-central conjugacy classes of G, and vertices K_1 and K_2 are adjacent if and only if there exists elements $g_1 \in K_1$ and $g_2 \in K_2$ such that $g_1 g_2 = g_2 g_1$. In this paper, we obtain the number of connected components of $\Gamma(S_n)$ and give an upper bound on the diameter of $\Gamma(S_n)$. We further prove that $\Gamma(S_n)$ is perfect if and only if $n \leq 7$ or $n = 9$ and conjecture that $\Gamma(S_n)$ is weakly perfect for all n.

1 Introduction

There are many graphs associated to groups. One such commonly studied graph is the commuting graph of a group G, where the vertex set is the elements of G and two vertices are adjacent if and only if the corresponding elements commute. In 2009, Herzog, Longobardi, and Maj [2] introduced the commuting conjugacy class of a group G as the graph with vertices being the non-identity conjugacy classes of G, and vertices K_1 and K_2 are adjacent if and only if there exists elements $g_1 \in K_1$ and $g_2 \in K_2$ such that $g_1 g_2 = g_2 g_1$; in this case we say K_1 and K_2 commute. Following the convention of the more recent papers, we define the commuting conjugacy class graph $\Gamma(G)$ of G to be the same graph as defined in [2], with the only difference being, we take the vertex set to be the non-central conjugacy classes of G.

More recently, there have been advances in the study of the commuting conjugacy class graphs. In particular, Mohammadian et. al. [3] classified all G such that $\Gamma(G)$ is triangle-free and Salahshour, Ashrafi proved that the graphs of finite CA-groups (i.e., a group in which the centralizer of any non-identity element is an abelian subgroup) are a union of complete graphs [5]. The aim of this paper is to describe the connected components of $\Gamma(S_n)$, as well as to investigate its diameter, clique number, and chromatic number. Even though most of the proofs in this paper don't rely on computational algebra packages, Sage has been extremely useful in verifying some of the conjectures for small values of n.

2 Connected Components of $\Gamma(S_n)$

In order to describe the connected components of $\Gamma(G)$ we will use a result about prime graphs. For a group G, the prime graph of G is the graph $\Pi(G)$ having vertices as the prime divisors of $|G|$, and vertices p and q are adjacent if and only if G contains an element of order pq. In [2], it was proved that $\Gamma(G)$ and $\Pi(G)$ have the same number of connected components.

Theorem 2.1. *Let p be a prime. If $n \in \{p, p+1\}$, then $\Gamma(S_n)$ has two connected components, of which one is an isolated vertex corresponding the conjugacy class of p-cycles. Otherwise, $\Gamma(S_n)$ is connected.*

Proof. It is easy to see that $\Gamma(S_3)$ has two connected components. Suppose $4 \leq n \in \{p, p+1\}$ for a prime p. Consider the prime graph $\Pi(S_n)$ of S_n. In either case, p is an isolated vertex and all the other vertices are adjacent to 2 as there cannot be large consecutive primes. Hence, $\Pi(S_n)$ has two connected components and so does $\Gamma(S_n)$. We now show that one of the connected components of $\Gamma(S_n)$ is an isolated vertex.

Case 1. Suppose $n = p$ for a prime p. Let $g \in S_n$ be an n-cycle. Then $|\mathrm{Cl}_{S_n}(g)| = (n-1)!$ and $|C_{S_n}(g)| = n$. Since n is prime, all elements of $\langle g \rangle$ are n-cycles. Hence, $C_{S_n}(g) = \langle g \rangle$ and the class of n-cycles is an isolated vertex.

Case 2. Suppose $n = p + 1$ for a prime p. Let $g \in S_n$ be an $(n-1)$-cycle. Then $|\mathrm{Cl}_{S_n}(g)| = n(n-2)!$ and $|C_{S_n}(g)| = n - 1$. Since $n - 1$ is prime, all elements of $\langle g - 1 \rangle$ are $(n-1)$-cycles. Hence, $C_{S_n}(g) = \langle g \rangle$ and the class of $(n-1)$-cycles is an isolated vertex.

Otherwise, suppose $n \notin \{p, p+1\}$. Then, n is at least 2 more than a prime; so, 2 is adjacent to all the other vertices of $\Pi(G)$. Hence, $\Gamma(S_n)$ is connected.

□

Remark. The description of isolated vertices of $\Gamma(S_n)$ is already seen in [4].

3 Diameter of $\Gamma(S_n)$

Let $\mathscr{P}(n)$ the set of partitions of $n \in \mathbb{N}$. In [1], Britnell and Wildon define the coarsenings of partitions as follows.

Definition 3.1. Given two partitions μ and ν in $\mathcal{P}(n)$, we say that ν is a coarsening of μ if ν can be obtained from μ by adding together parts of μ of the same size.

For example, $\left(4, 3^4, 1^2\right)$ has both $(12, 4, 2)$ and $\left(6, 4, 3^2, 1^2\right)$ as coarsenings. By definition, every partition is a coarsening of itself. The following theorem is Proposition 4 in [1], which shows that $\Gamma(S_n)$ can be constructed purely combinatorially. Notationwise, C^λ denotes the conjugacy class with representative λ.

Theorem 3.1. *Let $\lambda, \mu \in \mathscr{P}(n)$. The conjugacy classes C^λ and C^μ commute if and only if there is a partition $v \in \mathscr{P}(n)$ which is a coarsening of both λ and μ.*

In the above example, the only partition coarser than $(12, 4, 2)$ is itself and $(12, 4, 2)$ is not a coarsening of $\left(4, 3^4, 1^2\right)$; hence, the two conjugacy classes don't commute. Furthermore, the relation of being coarser is not transitive. For instance, $(12, 4, 2)$ is coarser than $(6^2, 4, 2)$, and $(6^2, 4, 2)$ is coarser than $(6, 4, 3^2, 1^2)$. However, $(12, 4, 2)$ is not coarser than $(6, 4, 3^2, 1^2)$.

We write $K_1 \succeq K_2$ if the corresponding partition for the cycle notation of K_1 is coarser than the corresponding partition for the cycle notation of K_2.

The *eccentricity* of a vertex u is the greatest distance between u and any other vertex. Let $\epsilon(K)$ denote the eccentricity of the vertex K.

Theorem 3.2. *Let p be a prime. If $n \notin \{p, p+1\}$ then $\epsilon(2^1, 1^{n-2}) \leq 3$ in $\Gamma(S_n)$.*

Proof. Let K_1 be an arbitrary conjugacy class of S_n and let $\operatorname{lpf}(n)$ denote the least prime factor of n.

Case 1. Suppose the conjugacy class K_1 contains a cycle of length $k > 1$ where $k \notin \{n, n-1\}$. Then K_1 is adjacent to $K_2 := (k^1, 1^{n-k})$ since $K_1 \succeq K_2$ and K_2 is adjacent to $(2^1, 1^{n-2})$ since $k+2 \leq n$ guarantess that we can find disjoint elements from the two conjugacy classes.

Case 2. Suppose the conjugacy class K_1 contains a cycle of length $k = n$. Let $p := \operatorname{lpf}(k)$ so that $k = mp$ for some $m \in \mathbb{Z}$. Then, K_1 is adjacent to $K_2 := (p^m)$ since $K_1 \succeq K_2$. Further, K_2 is adjacent to $K_3 := (p^1, 1^{n-p})$ since $K_2 \geq K_3$ and K_3 is adjacent to $(2^1, 1^{n-2})$ since $\operatorname{lpf}(n) + 2 \leq n$. This holds since n is composite.

Case 3. Suppose the conjugacy class K_1 contains a cycle of length $k = n-1$. Let $p := \operatorname{lpf}(k)$ so that $k = mp$ for some $m \in \mathbb{Z}$. Then, K_1 is adjacent to $K_2 := (p^m, 1^1)$ since $K_1 \succeq K_2$. Further, K_2 is adjacent to $K_3 := (p^1, 1^{n-p})$ and K_3 is adjacent to $(2^1, 1^{n-2})$ since $\operatorname{lpf}(n-1) + 2 \leq n$. This holds since $n-1$ is composite.

□

The *diameter* of a graph is the maximum eccentricity of any vertex that belongs to the graph.

Corollary. *For $n \notin \{p, p+1\}$, the diameter of $\Gamma(S_n)$ is ≤ 5.*

Proof. To form a path between any vertex to another, we can use $(2^1, 1^{n-2})$ as an intermediary vertex. For even n, *Case 2* reduces to two steps and for odd n, *Case 3* reduces to two steps. Therefore, in total, the distance between any two vertices at most 5. $\square$

Let $\Gamma^*(S_n)$ denote the larger connected component of $\Gamma(S_n)$.

Theorem 3.3. *Let p be a prime. For $n \in \{p, p+1\}$, we have $\epsilon(2^1, 1^{n-2}) \leq 2$ in $\Gamma^*(S_n)$.*

Proof. If $n \in \{p, p+1\}$, the class of p-cycles is isolated. If $n = p$, then $p-1$ is even, and the same argument as in *Case 1* and *Case 3* (now reduced to two steps) in the proof of Theorem 3.2 gives the result. Similarly, if $n = p + 1$, then $p + 1$ is even, and the same argument as in *Case 1* and *Case 2* (now reduced to two steps) in the proof of Theorem 3.2 gives the result. $\square$

Corollary. *For $n \in \{p, p + 1\}$, the diameter of $\Gamma^*(S_n)$ is ≤ 4.*

Let $\operatorname{diam}^*(\Gamma)$ denote the diameter of the largest connected component of Γ.

Conjecture 3.1. *$diam^*(\Gamma(S_n)) = 4$ for all $n \geq 8$.*

Remark. See the Appendix for values of $\operatorname{diam}^*(\Gamma(S_n))$ for small n.

4 Clique Number and Chromatic Number of $\Gamma(S_n)$

A *clique* of a graph G is a complete subgraph of G, and the *clique number* is the size of the largest clique of G. The *chromatic number* of a graph G is the smallest amount of colors to color the vertices of G so that adjacent vertices have different colors. Clearly, the chromatic number is always greater than or equal to the clique number of a graph. Let ω_n denote the clique number of $\Gamma(S_n)$ and χ_n denote the chromatic number of $\Gamma(S_n)$.

Theorem 4.1. *$\omega_n \geq \frac{n}{2}$ for even n and $\omega_n \geq \frac{n-1}{2}$ for odd n.*

Proof. The conjugacy classes of $S_{n/2}$ viewed as conjugacy classes of S_n form a complete subgraph $K_{n/2}$ of $\Gamma(S_n)$. To see this, let K_1 and K_2 be two such conjugacy classes. We can find $\tau \in K_1$ such that all entries in the cycle notation of τ are from the set $\{1, 2, \ldots, \frac{n}{2}\}$ and $\sigma \in K_2$ such that all entries in the cycle notation of τ are from the set $\{\frac{n}{2}, \frac{n}{2} + 1, \ldots, n\}$. The result follows since τ and σ are disjoint cycles, hence commute.

$\square$

Corollary. *$\chi_n \geq \frac{n}{2}$ for even n and $\chi_n \geq \frac{n-1}{2}$ for odd n.*

An *induced subgraph* of a graph G is a graph $S = (V', E')$ such that $V' \subseteq V$ and two vertices v and v' are adjacent in S if and only if the vertices v and v' are adjacent in G.

A graph is called *weakly perfect* if its chromatic number equals its clique number. If a graph G and all of its induced subgraphs are weakly perfect, then G is called *perfect*.

Computations (see the Appendix) suggest the following conjecture.

Conjecture 4.1. *$\Gamma(S_n)$ is weakly perfect for all n.*

Using Sage, we were able to verify Conjecture 4.1 computationally for all $n \leq 10$.

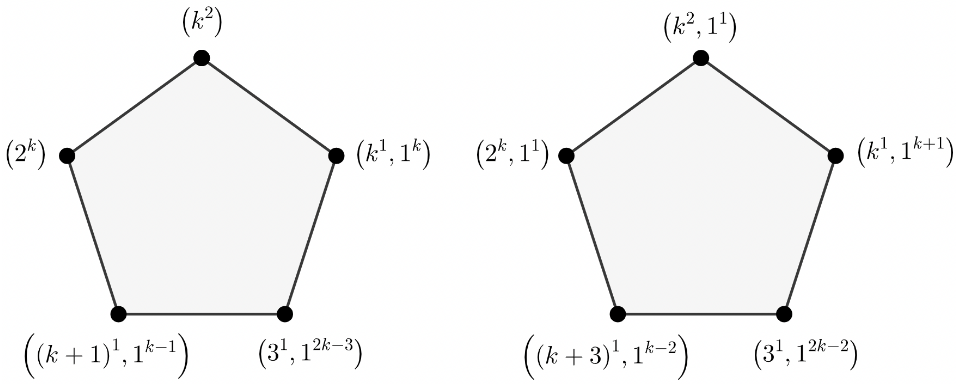

Figure 1: Lemma 4.3 and Lemma 4.4

Theorem 4.2. *$\Gamma(S_n)$ is perfect if and only if $n \leq 7$ or $n = 9$.*

The fact that $\Gamma(S_n)$ is perfect if $n \leq 7$ or $n = 9$ is verified computationally via Sage, which can be found in the Appendix. For all the remaining values of n, we will show that $\Gamma(S_n)$ has an induced subgraph isomorphic to a cycle of order 5, which implies that $\Gamma(S_n)$ is not perfect.

Lemma 4.3. *Let $n \geq 8$ satisfy $n \equiv 2 \pmod 4$. Then, $\Gamma(S_n)$ is not perfect.*

Proof. Let $n = 2k$ with $k > 3$ odd. We claim that the induced subgraph in $\Gamma(S_n)$ on the vertices

$$\{(k^2), (k^1, 1^k), (3^1, 1^{2k-3}), ((k+1)^1, 1^{k-1}), (2^k)\}$$

is a 5-cycle.

- (k^2) and $(k^1, 1^k)$ commute since $(k^2) \succeq (k^1, 1^k)$.
- $(k^1, 1^k)$ and $(3^1, 1^{2k-3})$ commute since $(k^1, 3^1, 1^{k-3}) \succeq (k^1, 1^k)$ and $(k^1, 3^1, 1^{k-3}) \succeq (3^1, 1^{2k-3})$.

- $(3^1, 1^{2k-3})$ and $((k+1)^1, 1^{k-1})$ commute since $((k+1)^1, 3^1, 1^{k-4}) \succeq (3^1, 1^{2k-3})$ and $((k+1)^1, 3^1, 1^{k-4}) \succeq ((k+1)^1, 1^{k-1})$.
- $((k+1)^1, 1^{k-1})$ and (2^k) commute since $((k+1)^1, (k-1)^1) \succeq ((k+1)^1, 1^{k-1})$ and $((k+1)^1, (k-1)^1) \succeq (2^k)$.
- (2^k) and (k^2) commute since $((2k)^1) \succeq (k^2)$ and $((2k)^1) \succeq (2^k)$.

We now show that these were all the edges.

- (k^2) and $(3^1, 1^{2k-3})$ don't commute since $(3^1) \in K$ for any partition $K \succeq (3^1, 1^{2k-3})$, but then $K \not\succeq (k^2)$.
- (k^2) and $((k+1)^1, 1^{k-1})$ don't commute since $(k+1)^1 \in K$ for any partition $K \succeq ((k+1)^1, 1^{k-1})$, but then $K \not\succeq (k^2)$.
- $(k^1, 1^k)$ and $((k+1)^1, 1^{k-1})$ don't commute since $(k+1)^1 \in K$ for any partition $K \succeq ((k+1)^1, 1^{k-1})$, and $k^1 \in K$ for any partition $K \succeq (k^1, 1^k)$, but such K cannot exist as $k + (k+1) > 2k$.
- $(k^1, 1^k)$ and (2^k) don't commute since $k^1 \in K$ for any partition $K \succeq (k^1, 1^k)$, but then $K \not\succeq (2^k)$.
- $(3^1, 1^{2k-3})$ and (2^k) don't commute since $3^1 \in K$ for any partition $K \succeq (3^1, 1^{2k-3})$, but then $K \not\succeq (2^k)$.

□

Lemma 4.4. *Let $n \geq 8$ satisfy $n \equiv 3 \pmod 4$. Then, $\Gamma(S_n)$ is not perfect.*

Proof. Let $n = 2k+1$ with $k > 3$ odd. We claim that the induced subgraph in $\Gamma(S_n)$ on the vertices

$$\{(k^2, 1), (k^1, 1^{k+1}), (3^1, 1^{2k-2}), ((k+3)^1, 1^{k-2}), (2^k, 1^1)\}$$

is a 5-cycle.

- $(k^2, 1)$ and $(k^1, 1^{k+1})$ commute since $(k^2, 1) \succeq (k^1, 1^{k+1})$.
- $(k^1, 1^{k+1})$ and $(3^1, 1^{2k-2})$ commute since $(k^1, 3^1, 1^{k-2}) \succeq (k^1, 1^k)$ and $(k^1, 3^1, 1^{k-2}) \succeq (3^1, 1^{2k-2})$.
- $(3^1, 1^{2k-2})$ and $((k+3)^1, 1^{k-2})$ commute since $((k+3)^1, 3^1, 1^{k-5}) \succeq (3^1, 1^{2k-2})$ and $((k+3)^1, 3^1, 1^{k-5}) \succeq ((k+3)^1, 1^{k-2})$.
- $((k+3)^1, 1^{k-2})$ and $(2^k, 1^1)$ commute since $((k+3)^1, (k-3)^1, 1^1) \succeq ((k+3)^1, 1^{k-2})$ and $((k+3)^1, (k-3)^1, 1^1) \succeq (2^k, 1^1)$.
- $(2^k, 1)$ and $(k^2, 1)$ commute since $((2k)^1, 1^1) \succeq (k^2, 1)$ and $((2k)^1, 1^1) \succeq (2^k, 1)$.

We now show that these were all the edges.

- $(k^2, 1)$ and $(3^1, 1^{2k-2})$ don't commute since $(3^1) \in K$ for any partition $K \succeq (3^1, 1^{2k-2})$, but then $K \not\succeq (k^2, 1)$.
- $(k^2, 1)$ and $((k+3)^1, 1^{k-2})$ don't commute since $(k+3)^1 \in K$ for any partition $K \succeq ((k+3)^1, 1^{k-2})$, but then $K \not\succeq (k^2, 1)$.
- $(k^1, 1^{k+1})$ and $((k+3)^1, 1^{k-2})$ don't commute since $(k+3)^1 \in K$ for any partition $K \succeq ((k+3)^1, 1^{k-2})$, and $k^1 \in K$ for any partition $K \succeq (k^1, 1^{k+1})$, but such K cannot exist as $k + (k+3) > 2k+1$.
- $(k^1, 1^{k+1})$ and $(2^k, 1^1)$ don't commute since $k^1 \in K$ for any partition $K \succeq (k^1, 1^{k+1})$, but then $K \not\succeq (2^k, 1^1)$.
- $(3^1, 1^{2k-2})$ and $(2^k, 1^1)$ don't commute since $3^1 \in K$ for any partition $K \succeq (3^1, 1^{2k-2})$, but then $K \not\succeq (2^k, 1^1)$.

□

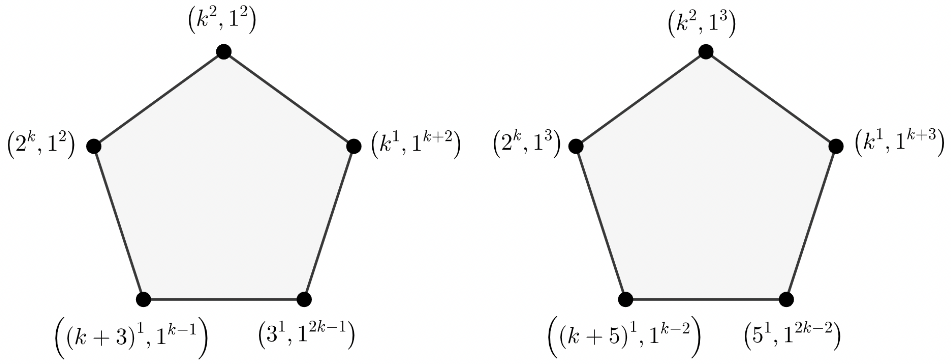

Figure 2: Lemma 4.5 and Lemma 4.6

Lemma 4.5. *Let $n \geq 8$ satisfy $n \equiv 0 \pmod 4$. Then, $\Gamma(S_n)$ is not perfect.*

Proof. If $n = 8$, then it is easy to verify that the induced subgraph in $\Gamma(S_8)$ on the vertices

$$\{(2^4), (3^1, 2^1, 1^3), (3^1, 2^2, 1^1), (3^2, 1^2), (4^1, 1^4)\}$$

is a 5-cycle. Let $n = 2k + 2$ with $k > 3$ odd. We claim that the induced subgraph in $\Gamma(S_n)$ on the vertices

$$\{(k^2, 1^2), (k^1, 1^{k+2}), (3^1, 1^{2k-1}), ((k+3)^1, 1^{k-1}), (2^k, 1^2)\}$$

is a 5-cycle.

- $(k^2, 1^2)$ and $(k^1, 1^{k+2})$ commute since $(k^2, 1^2) \succeq (k^1, 1^{k+2})$.
- $(k^1, 1^{k+2})$ and $(3^1, 1^{2k-1})$ commute since $(k^1, 3^1, 1^{k-1}) \succeq (k^1, 1^{k+2})$ and $(k^1, 3^1, 1^{k-1}) \succeq (3^1, 1^{2k-1})$.
- $(3^1, 1^{2k-1})$ and $((k+3)^1, 1^{k-1})$ commute since $((k+3)^1, 3^1, 1^{k-4}) \succeq (3^1, 1^{2k-1})$ and $((k+3)^1, 3^1, 1^{k-4}) \succeq ((k+3)^1, 1^{k-1})$.
- $((k+3)^1, 1^{k-1})$ and $(2^k, 1^2)$ commute since $((k+3)^1, (k-3)^1, 1^2) \succeq ((k+3)^1, 1^{k-1})$ and $((k+3)^1, (k-3)^1, 1^2) \succeq (2^k, 1^2)$.
- $(2^k, 1^2)$ and $(k^2, 1^2)$ commute since $((2k)^1, 1^2) \succeq (k^2, 1^2)$ and $((2k)^1, 1^2) \succeq (2^k, 1^2)$.

We now show that these were all the edges.

- $(k^2, 1^2)$ and $(3^1, 1^{2k-1})$ don't commute since $(3^1) \in K$ for any partition $K \succeq (3^1, 1^{2k-1})$, but then $K \not\succeq (k^2, 1^2)$.
- $(k^2, 1^2)$ and $((k+3)^1, 1^{k-1})$ don't commute since $(k+3)^1 \in K$ for any partition $K \succeq ((k+3)^1, 1^{k-1})$, but then $K \not\succeq (k^2, 1^2)$.
- $(k^1, 1^{k+2})$ and $((k+3)^1, 1^{k-1})$ don't commute since $(k+3)^1 \in K$ for any partition $K \succeq ((k+3)^1, 1^{k-1})$, and $k^1 \in K$ for any partition $K \succeq (k^1, 1^{k+2})$, but such K cannot exist as $k + (k+3) > 2k+2$.
- $(k^1, 1^{k+2})$ and $(2^k, 1^2)$ don't commute since $k^1 \in K$ for any partition $K \succeq (k^1, 1^{k+2})$, but then $K \not\succeq (2^k, 1^2)$.
- $(3^1, 1^{2k-1})$ and $(2^k, 1^2)$ don't commute since $3^1 \in K$ for any partition $K \succeq (3^1, 1^{2k-1})$, but then $K \not\succeq (2^k, 1^2)$.

□

Lemma 4.6. *Let $n \geq 8$ satisfy $n \equiv 1 \pmod 4$. Then, $\Gamma(S_n)$ is not perfect.*

Proof. If $n = 13$, then it is easy to verify that the induced subgraph in $\Gamma(S_{13})$ on the vertices

$$\{(2^1, 1^{11}), (2^3, 1^7), (3^4, 1^1), (4^3, 1^1), (8^1, 1^5)\}$$

is a 5-cycle. Let $n = 2k+3$ with $k > 5$ odd. We claim that the induced subgraph in $\Gamma(S_n)$ on the vertices

$$\{(k^2, 1^3), (k^1, 1^{k+3}), (5^1, 1^{2k-2}), ((k+5)^1, 1^{k-2}), (2^k, 1^3)\}$$

is a 5-cycle.

- $(k^2, 1^3)$ and $(k^1, 1^{k+3})$ commute since $(k^2, 1^3) \succeq (k^1, 1^{k+3})$.

- $(k^1, 1^{k+3})$ and $(5^1, 1^{2k-2})$ commute since $(k^1, 5^1, 1^{k-2}) \succeq (k^1, 1^{k+3})$ and $(k^1, 5^1, 1^{k-2}) \succeq (5^1, 1^{2k-2})$.
- $(5^1, 1^{2k-2})$ and $((k+5)^1, 1^{k-2})$ commute since $((k+5)^1, 5^1, 1^{k-7}) \succeq (5^1, 1^{2k-2})$ and $((k+5)^1, 5^1, 1^{k-7}) \succeq ((k+5)^1, 1^{k-2})$.
- $((k+5)^1, 1^{k-2})$ and $(2^k, 1^3)$ commute since $((k+5)^1, (k-5)^1, 1^3) \succeq ((k+5)^1, 1^{k-2})$ and $((k+5)^1, (k-5)^1, 1^3) \succeq (2^k, 1^3)$.
- $(2^k, 1^3)$ and $(k^2, 1^3)$ commute since $((2k)^1, 1^3) \succeq (k^2, 1^3)$ and $((2k)^1, 1^3) \succeq (2^k, 1^3)$.

We now show that these were all the edges.

- $(k^2, 1^3)$ and $(5^1, 1^{2k-2})$ don't commute since $(5^1) \in K$ for any partition $K \succeq (5^1, 1^{2k-2})$, but then $K \not\succeq (k^2, 1^3)$.
- $(k^2, 1^3)$ and $((k+5)^1, 1^{k-2})$ don't commute since $(k+5)^1 \in K$ for any partition $K \succeq ((k+5)^1, 1^{k-2})$, but then $K \not\succeq (k^2, 1^3)$.
- $(k^1, 1^{k+3})$ and $((k+5)^1, 1^{k-2})$ don't commute since $(k+5)^1 \in K$ for any partition $K \succeq ((k+5)^1, 1^{k-2})$, and $k^1 \in K$ for any partition $K \succeq (k^1, 1^{k+3})$, but such K cannot exist as $k + (k+5) > 2k+3$.
- $(k^1, 1^{k+3})$ and $(2^k, 1^3)$ don't commute since $k^1 \in K$ for any partition $K \succeq (k^1, 1^{k+3})$, but then $K \not\succeq (2^k, 1^3)$.
- $(5^1, 1^{2k-2})$ and $(2^k, 1^3)$ don't commute since $5^1 \in K$ for any partition $K \succeq (5^1, 1^{2k-2})$, but then $K \not\succeq (2^k, 1^3)$.

□

Therefore, we have established Theorem 4.2.

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Appendix: Jupyter Notebook for Sage Computations

We first implement a function which returns all the coarsening of a partition.

```
def coarsenings(partition):
    """
    Generate all coarsenings of a partition.
    """
    # if partition is empty, return itself
    if not partition:
        return [()]

    # Sort the partition for easier handling
    partition = sorted(partition, reverse=True)

    first = partition[0]
    count = partition.count(first)

    # If there's only one of the first element, we'll consider its coarsenings
    if count == 1:
        smaller_coarsenings = coarsenings(partition[1:])
        return [tuple(sorted((first,) + coarsening, reverse=True)) for␣
↪coarsening in smaller_coarsenings]

    # Otherwise, we'll split it into combinations
    results = []
    for i in range(1, count + 1):
        next_coarsenings = coarsenings(partition[i:])
        combined = first * i
        results.extend([tuple(sorted((combined,) + coarsening, reverse=True))␣
↪for coarsening in next_coarsenings])

    # Remove duplicates
    return list(set(results))
```

We can now construct $\Gamma(S_n)$ using **Theorem 3.1** and the built-in *Partitions* function of Sage.

```
def ccc_graph(n):
    """
    Create a graph where vertices are partitions of n (except the trivial␣
↪one) and two vertices are adjacent
    iff they have a common coarsening.
    """
    # Initialise a new graph
    G = Graph()

    # Generate all partitions of n except the trivial one
    partitions = [p for p in Partitions(n) if not all(val == 1 for val in p)]

    # Precompute coarsenings for each partition and store in a dictionary
```

```
    partition_coarsenings = {p: set(coarsenings(p)) for p in partitions}

    # Add vertices to the graph
    G.add_vertices(partitions)

    part_len = len(partitions)

    # Check each pair of partitions for a common coarsening using the␣
↪precomputed data
    for i in range(part_len):
        for j in range(i + 1, part_len):
            if partition_coarsenings[partitions[i]].
↪intersection(partition_coarsenings[partitions[j]]):
                G.add_edge(partitions[i], partitions[j])

    return G
```

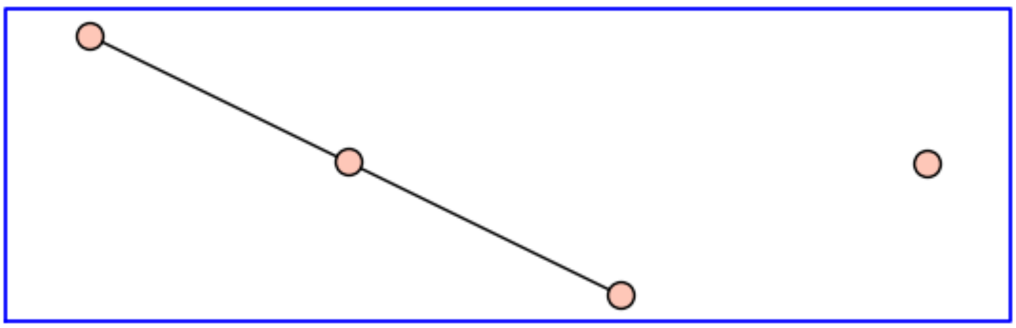
We can now store $\Gamma(S_n)$ as grph'n'.

```
# if you increase the range to i>20 the computations take a while to complete
for i in range(3, 21):
    grph = ccc_graph(i)
    globals()[f'grph{i}'] = grph
```

We now plot $\Gamma(S_n)$ for small values of n.

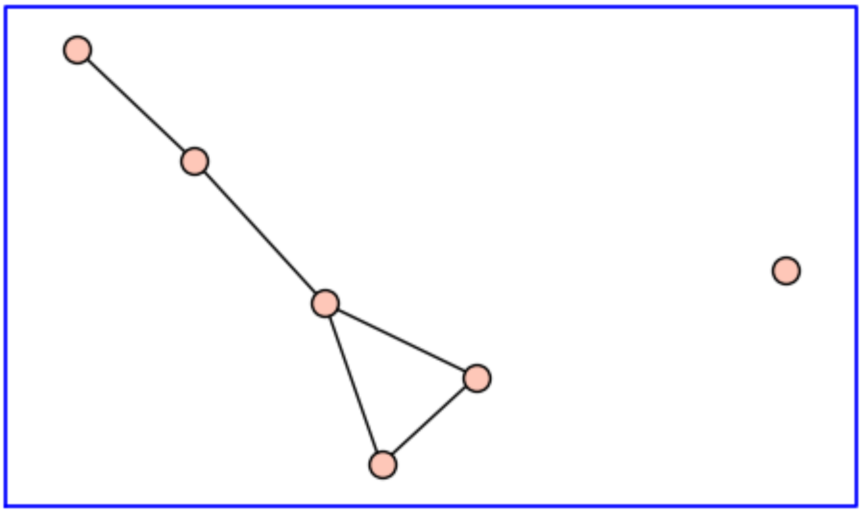
```
grph4.show(figsize=(10, 2), vertex_size = 100, graph_border=True,␣
↪vertex_labels=False)
```

$\Gamma(S_4)$



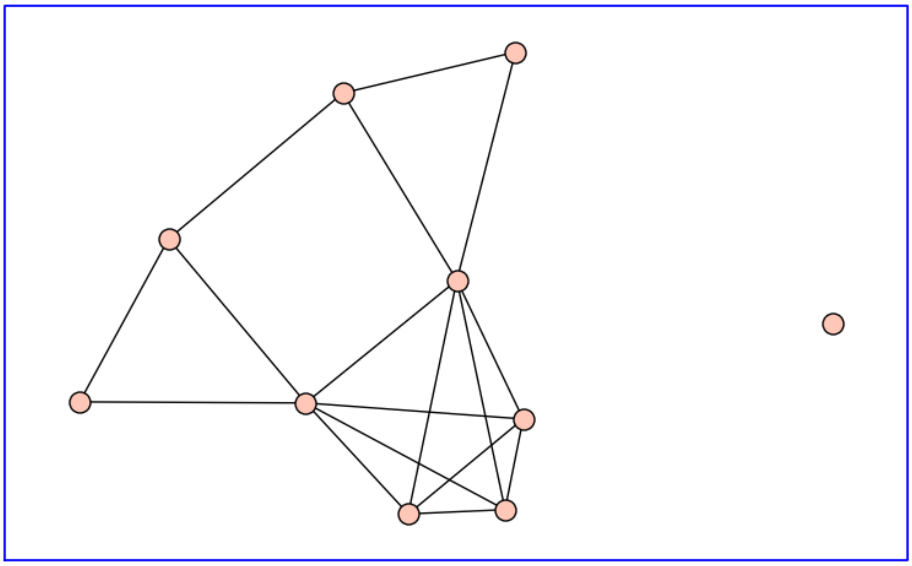
```
grph5.show(figsize=(10, 3), vertex_size = 100, graph_border=True,␣
↪vertex_labels=False, layout = "spring")
```

$\Gamma(S_5)$



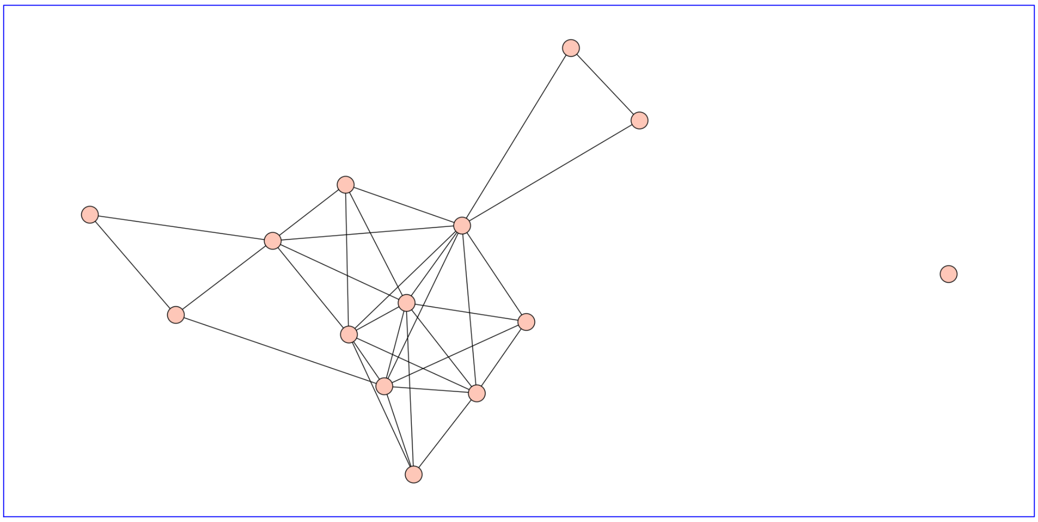
```
grph6.show(figsize=(15, 5), vertex_size = 150, graph_border=True,␣
↪vertex_labels=False)
```

$\Gamma(S_6)$



```
grph7.show(figsize=(20, 20), vertex_size = 500, graph_border=True,␣
↪vertex_labels=False,  layout = "spring")
```

$\Gamma(S_7)$



We now write a function to compute $\mathrm{diam}^*(\Gamma(S_n))$.

```
def diam(G):
    # Decompose the graph into its connected components
    components = G.connected_components_subgraphs()

    # Find the largest component by number of vertices
    largest_component = max(components, key=lambda H: H.num_verts())

    # Compute and return the diameter of the largest component
    return largest_component.diameter()
```

We can now observe computational evidence to support **Conjecture 3.1** for small n.

```
header = 'n' + " " * 9  # Initial spacing for the index labels
for i in range(4, 21):
    header += str(i).rjust(4)

print(header)

row = "Diameter*".ljust(10)
for i in range(4, 21):
    row += str(diam(globals()['grph' + str(i)])).rjust(4)

print(row)
```

```
n            4   5   6   7   8   9  10  11  12  13  14  15  16  17  18  19  20
Diameter*    2   3   3   3   4   4   4   4   4   4   4   4   4   4   4   4   4
```

We can also observe evidence to support **Conjecture 4.1** and **Theorem 4.2** for small n.

```
# Note that computing the chromatic number is computationally much harder␣
↪than computing the ccc-graph and its clique number
print("n".ljust(4), "Chromatic Number", "Clique Number")
for i in range(3,11):
    grph = globals()['grph' + str(i)]
    print(str(i).ljust(13), str(grph.chromatic_number()).ljust(13), grph.
↪clique_number())
```

```
n    Chromatic Number Clique Number
3             1             1
4             2             2
5             3             3
6             5             5
7             5             5
8             7             7
9             11            11
10            11            11
```

```
print("n".ljust(4), "Is Perfect?")
for i in range(3,15):
    grph = globals()['grph' + str(i)]
    print(str(i).ljust(8), grph.is_perfect())
```

```
n    Is Perfect?
3        True
4        True
5        True
6        True
7        True
8        False
9        True
10       False
11       False
12       False
13       False
14       False
```